Dual embeddings of dense near polygons

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Abstract

Let $e : \mathcal{S} \rightarrow \Sigma$ be a full polarized projective embedding of a dense near polygon $\mathcal{S}$, i.e., for every point p of $\mathcal{S}$, the set H_p of points at non-maximal distance from p is mapped by e into a hyperplane Π_p of Σ. We show that if every line of $\mathcal{S}$ is incident with precisely three points or if $\mathcal{S}$ satisfies a certain property (P_{de}) then the map $p \mapsto \Pi_p$ defines a full polarized embedding e^* (the so-called dual embedding of e) of $\mathcal{S}$ into a subspace of the dual Σ^* of Σ. This generalizes a result of [6] where it was shown that every embedding of a thick dual polar space has a dual embedding. We determine which known dense near polygons satisfy property (P_{de}). This allows us to conclude that every full polarized embedding of a known dense near polygon has a dual embedding.

Keywords: near polygon, polarized embedding, dual embedding, hyperplane
MSC2000: 51A45, 51A50, 51E12, 05B25

1 Introduction

A *near polygon* is a connected partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$, satisfying the property that for every point $p \in \mathcal{P}$ and every line $L \in \mathcal{L}$, there exists a unique point $\pi_L(p)$ on L nearest to p. Here distances are measured in the collinearity graph Γ of $\mathcal{S}$. If d is the diameter of Γ, then the near polygon is called a *near* $2d$*-gon*. A near 0-gon is just a point and a near 2-gon is a line. Near quadrangles are usually called generalized quadrangles (GQ's). We refer to Payne and Thas [10] for the basic notions on generalized quadrangles to be used in this paper.

A near polygon is called *slim* if every line is incident with precisely three points. A near polygon is called *dense* if every line is incident with

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at least three points and if every two points at distance 2 have at least two common neighbours. If x and y are two points of a dense near polygon at distance k from each other, then by Theorem 4 of Brouwer and Wilbrink [4], x and y are contained in a unique convex subspace of diameter k. These convex subspaces are called *quads* if $k = 2$ and *hexes* if $k = 3$. The points and lines contained in a convex subspace of diameter k define a sub-near-$2k$-gon. Convex subspaces of diameter k are therefore also called *convex sub-$2k$-gons*. We will now introduce two properties of dense near polygons.

(I) Let $\mathcal{S}$ be a dense near $2n$-gon, $n \geq 2$, let x be a point of $\mathcal{S}$, let B be a convex sub-$2(n-2)$-gon through x and let A be a convex sub-$2(n-1)$-gon through B. Define the following graph $\Gamma(x, B, A)$:

- the vertices of $\Gamma(x, B, A)$ are the convex subspaces of diameter $n-1$ through B distinct from A;
- two distinct vertices A_1 and A_2 of $\Gamma(x, B, A)$ are adjacent if there exists a quad Q satisfying (i) $Q \cap B$ is a point at distance $n-2$ from x, (ii) $Q \cap A$, $Q \cap A_1$ and $Q \cap A_2$ are lines.

We say that $\mathcal{S}$ satisfies *property* (P_{de}) if the graph $\Gamma(x, B, A)$ is connected for all points x, for all convex subspaces B of diameter $n-2$ and all convex subspaces A of diameter $n-1$ satisfying $\{x\} \subseteq B \subseteq A$.

(II) We say that a dense near polygon $\mathcal{S}$ satisfies *property* (P'_{de}) if for every convex subspace B of diameter $n-2$, for every point x of B and for every three distinct convex subspaces A_1, A_2 and A_3 of diameter $n-1$ through B, there exists a quad Q satisfying (i) $Q \cap B = \{x\}$, (ii) $Q \cap A_i$ is a line for every $i \in \{1, 2, 3\}$. A dense near polygon which satisfies property (P'_{de}) also satisfies property (P_{de}).

Let $\mathcal{S}$ be a partial linear space. A *hyperplane* of $\mathcal{S}$ is a proper subspace meeting each line of $\mathcal{S}$. A *full (projective) embedding* of $\mathcal{S}$ is an injective mapping e from the point-set $\mathcal{P}$ of $\mathcal{S}$ to the point-set of a projective space Σ satisfying (i) $\langle e(\mathcal{P}) \rangle = \Sigma$ and (ii) $e(L) := \{e(x) \,|\, x \in L\}$ is a line of Σ for every line L of $\mathcal{S}$. If $e : \mathcal{S} \to \Sigma$ is a full embedding, then for every hyperplane Π of Σ, $e^{-1}(e(\mathcal{P}) \cap \Pi)$ is a hyperplane of $\mathcal{S}$. We say that the hyperplane $e^{-1}(e(\mathcal{P}) \cap \Pi)$ *arises* from e.

If $\mathcal{S}$ is a dense near $2n$-gon and x is a point of $\mathcal{S}$, then the set H_x of points at distance at most $n-1$ from x is a hyperplane of $\mathcal{S}$, called the *singular hyperplane* of $\mathcal{S}$ with *deepest point* x. [It follows from the theory of dense near polygons that H_x cannot coincide with the whole set of points of $\mathcal{S}$. If y is a point of $\mathcal{S}$ at maximal distance k from x, then the unique convex sub-$2k$-gon of $\mathcal{S}$ containing x and y must coincide with $\mathcal{S}$, see e.g. Theorem 2.14 of [8]. This forces k to be equal to n.] A full embedding $e : \mathcal{S} \to \Sigma$ of a dense near polygon $\mathcal{S}$ is called *polarized* if every *singular* *hyperplane* of $\mathcal{S}$ arises from e, or equivalently, if every singular hyperplane of $\mathcal{S}$ is mapped by e into a necessarily unique hyperplane of Σ.

The following is the main result of this paper.

Main Theorem. *Let $e : \mathcal{S} \to \Sigma$ be a full polarized embedding of a dense near $2n$-gon $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ $(n \geq 1)$. For every point p of $\mathcal{S}$, let $e^*(p)$ be the point $\langle e(H_p) \rangle$ of the dual Σ^* of Σ. Let $\Sigma^{(*)}$ be the subspace of Σ^* generated by all points $\langle e(H_p) \rangle$, $p \in \mathcal{P}$. If $\mathcal{S}$ is slim or if $\mathcal{S}$ satisfies property (P_{de}), then e^* defines a full polarized embedding of $\mathcal{S}$ in $\Sigma^{(*)}$.*

Definition. The embedding e^* defined in the Main Theorem is called the *dual embedding* of e.

The subscript "de" in property (P_{de}) refers to the word "dual embedding". The Main Theorem generalizes a result of Cardinali, De Bruyn and Pasini [6, Theorem 1.7] who showed that every full polarized embedding of a thick dual polar space admits a dual embedding. This fact also follows from our Main Theorem: in Section 5.1, we will show that every dual polar space with at least 3 points on every line satisfies property (P_{de}). Theorem 1.7 of [6] has already found applications in the study of the structure of full polarized embeddings of dual polar spaces, see Section 2 of Cardinali and De Bruyn [5]. Many dense near polygons admit a full embedding. This is certainly the case for all slim dense near polygons (see e.g. Proposition 3.11) and also for many examples of dual polar spaces and so-called product and glued near polygons.

Our paper is organized as follows. In Section 2, we will define additional notions regarding near polygons and embeddings of point-line geometries which we will use throughout this paper. In Section 3, we will consider the problem of the existence of dual embeddings for a more general class of point-line geometries. We will also prove there (see Proposition 3.11) that every full polarized embedding of a slim dense near polygon admits a dual embedding. In Section 4, we will extend this result to arbitrary full embeddings of dense near polygons satisfying property (P_{de}). In Section 5, we will show that every known dense near $2n$-gon, $n \geq 2$, satisfies property (P_{de}) except for the ones containing a so-called $\mathbb{E}_1$-hex or $\mathbb{E}_2$-hex. This shows that having property (P_{de}) is not an uncommon property. The whole discussion allows us to conclude that every full embedding of every known dense near polygon has a dual embedding.

2 Additional notions regarding near polygons and embeddings of point-line geometries

Let $\mathcal{S}$ be a given point-line geometry. Two full embeddings $e_1 : \mathcal{S} \to \Sigma_1$ and $e_2 : \mathcal{S} \to \Sigma_2$ of $\mathcal{S}$ are called *isomorphic* ($e_1 \cong e_2$) if there exists an isomorphism $f : \Sigma_1 \to \Sigma_2$ such that $e_2 = f \circ e_1$. If $e : \mathcal{S} \to \Sigma$ is a full embedding of $\mathcal{S}$ and if U is a subspace of Σ satisfying (C1) $\langle U, e(p)\rangle \neq U$ for every point p of $\mathcal{S}$, and (C2) $\langle U, e(p_1)\rangle \neq \langle U, e(p_2)\rangle$ for any two distinct points p_1 and p_2 of $\mathcal{S}$, then there exists a full embedding e/U of $\mathcal{S}$ in the quotient space Σ/U, mapping each point p of $\mathcal{S}$ to $\langle U, e(p)\rangle$. If $e_1 : \mathcal{S} \to \Sigma_1$ and $e_2 : \mathcal{S} \to \Sigma_2$ are two full embeddings, then we say that $e_1 \geq e_2$, if there exists a subspace U in Σ_1 satisfying (C1), (C2) and $e_1/U \cong e_2$. If $e : \mathcal{S} \to \Sigma$ is a full embedding of $\mathcal{S}$, then by Ronan [11], there exists up to isomorphism a unique full embedding $\widetilde{e} : \mathcal{S} \to \widetilde{\Sigma}$ satisfying the following: (i) $\widetilde{e} \geq e$; (ii) if $e' \geq e$ for some embedding e' of $\mathcal{S}$, then $\widetilde{e} \geq e'$. We say that $\widetilde{e}$ is *universal relative to* e. If $\widetilde{e'} \cong \widetilde{e}$ for any other embedding e' of $\mathcal{S}$ with the same underlying division ring, then $\widetilde{e}$ is called *absolutely universal*.

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a near polygon. We will denote the distance between two points x and y of $\mathcal{S}$ by $\mathrm{d}(x, y)$. For every point x of $\mathcal{S}$, for every nonempty subset X of $\mathcal{P}$ and every $i \in \mathbb{N}$, we define $\Gamma_i(x) = \{y \in \mathcal{P} \,|\, \mathrm{d}(x, y) = i\}$, $\mathrm{d}(x, X) = \min\{\mathrm{d}(x, y) \,|\, y \in X\}$ and $\Gamma_i(X) = \{y \in \mathcal{P} \,|\, \mathrm{d}(y, X) = i\}$. The maximal distance between two points of X is called the *diameter* of X and is denoted as $\mathrm{diam}(X)$. If X_1 and X_2 are two nonempty sets of points, then we define $\mathrm{d}(X_1, X_2) = \min\{\mathrm{d}(x_1, x_2) \,|\, x_1 \in X_1 \text{ and } x_2 \in X_2\}$. We will denote by $\langle *_1, *_2, \ldots, *_k\rangle$ the smallest convex subspace of $\mathcal{S}$ containing the objects $*_1, *_2, \ldots, *_k$. Here, $*_i$, $i \in \{1, \ldots, k\}$, can be either a point or a nonempty set of points of $\mathcal{S}$. Obviously, $\langle *_1, *_2, \ldots, *_k\rangle$ is the intersection of all convex subspaces of $\mathcal{S}$ containing $*_1, *_2, \ldots, *_k$.

Suppose now that $\mathcal{S}$ is a dense near $2n$-gon. If x is a point of $\mathcal{S}$, then the lines and quads through x define a linear space $\mathcal{L}(\mathcal{S}, x)$ called *the local space at* x. The convex sub-$2i$-gons, $i \in \{1, \ldots, n-1\}$, through x define a rank-$(n-1)$-geometry $\mathcal{G}(\mathcal{S}, x)$ which is called the *local geometry at* x. A convex sub-2δ-gon A of $\mathcal{S}$ is called *big* in $\mathcal{S}$ if every point of $\mathcal{S}$ outside A is collinear with a necessarily unique point of A. We then necessarily have $\delta = n - 1$. A convex subpolygon A is called *classical* in $\mathcal{S}$ if for every point x of $\mathcal{S}$, there exists a necessarily unique point $\pi_A(x)$ in A such that $\mathrm{d}(x, y) = \mathrm{d}(x, \pi_A(x)) + \mathrm{d}(\pi_A(x), y)$ for every point y of A. If A is big, then A is also classical in $\mathcal{S}$. If A is a classical convex sub-2δ-gon of $\mathcal{S}$ and if B is a convex sub-$2\delta'$-gon of $\mathcal{S}$ meeting A, then the diameter of $A \cap B$ is at least $\delta + \delta' - n$ by Theorem 2.32 of [8].

If H is a hyperplane of a dense near polygon $\mathcal{S}$ and if Q is a quad of $\mathcal{S}$, then either (i) $Q \subseteq H$, (ii) $Q \cap H$ consists of those points of Q which are collinear with a given point x of Q, (iii) $Q \cap H$ is a subquadrangle of Q, (iv) $Q \cap H$ is an *ovoid* of Q, i.e., a set of points of Q meeting each line in a unique point. If case (i), case (ii), case (iii), respectively case (iv), occurs, then we say that Q is *deep*, *singular*, *subquadrangular*, respectively *ovoidal*, with respect to H. If case (ii) occurs, then we call x the *deep point* of Q with respect to H.

3 Embeddings of a class of parapolar spaces

Definitions. (1) A partial linear space is called a *polar space* if for every point p and every line L, either 1 or all points of L are collinear with p. The *radical* of a polar space is the set of points collinear with all points. A polar space is called *nondegenerate* if its radical is empty. A subspace of a polar space is said to be *singular* if any two points of it are collinear. The rank r of a nondegenerate polar space is the maximal length r of a chain $S_0 \subset S_1 \subset \cdots \subset S_r$ of singular subspaces where $S_0 = \emptyset$ and $S_i \neq S_{i+1}$ for all $i \in \{0, \ldots, r-1\}$. A nondegenerate polar space of rank 2 is just a nondegenerate generalized quadrangle.

(2) A partial linear space is called a *gamma space* if for every point p and every line L, either 0, 1 or all points of L are collinear with p.

(3) A *parapolar space* is a connected partial linear gamma space possessing a collection of convex subspaces, called *symplecta*, isomorphic to nondegenerate polar spaces of rank at least 2, with the property that each line is contained in a symplecton and that each quadrangle is contained in a unique symplecton. A parapolar space in which every pair of points at distance 2 are contained in a necessarily unique symplecton is called a *strong parapolar space*.

Shult introduced in [13] a class $\mathcal{E}$ of parapolar spaces. The class $\mathcal{E}$ is equal to $\mathcal{E}_0 \cup \mathcal{E}_1 \cup \mathcal{E}_2 \cdots$, where the subclasses $\mathcal{E}_i$, $i \in \mathbb{N}$, are defined inductively in the following way. The subclass $\mathcal{E}_0$ contains one member, namely the point, and the subclass $\mathcal{E}_1$ consists of the lines which are incident with at least three points. The subclass $\mathcal{E}_n$, $n \geq 2$, contains those geometries $\mathcal{S}$ which satisfy the following properties:

(E_1) $\mathcal{S}$ is connected and its diameter is equal to n;

(E_2) every line of $\mathcal{S}$ is incident with at least three points;

(E_3) any geodesic in $\mathcal{S}$ completes to a geodesic of length n;

(E_4) for every point x of $\mathcal{S}$, the set H_x of points of $\mathcal{S}$ at distance at most $n-1$ from x is a hyperplane of $\mathcal{S}$;

(E_5) if x_1 and x_2 are two points of $\mathcal{S}$ with $k := \mathrm{d}(x_1, x_2) < n$, then the convex closure $\langle x_1, x_2 \rangle$ is a member of $\mathcal{E}_k$.

The elements of $\mathcal{E}_2$ are precisely the nondegenerate polar spaces in which each line is incident with at least three points. Every member of $\mathcal{E}_n$, $n \geq 2$, is a strong parapolar space. The symplecta are the convex closures of the pairs of points at distance 2 from each other. The class $\mathcal{E}$ contains every thick dual polar space and more generally every dense near polygon. The class $\mathcal{E}$ also contains some half-spin geometries, some Grassmann spaces and some exceptional geometries, see Shult [13, Section 6].

The geometric hyperplane H_x defined in (E_4) is called the *singular hyperplane* of $\mathcal{S}$ with *deepest point* x. By Shult [13, Lemma 6.1 (ii)], every geometric hyperplane of an element of $\mathcal{E}$ is a maximal subspace. In particular, this holds for the singular hyperplanes.

Definition. Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a partial linear space. A set $\mathcal{H}$ of hyperplanes of $\mathcal{S}$ is called a *pencil of hyperplanes* if $\bigcup_{H \in \mathcal{H}} H = \mathcal{P}$ and $H_1 \cap H_2 = H_1 \cap H_3 = H_2 \cap H_3$ for any three distinct hyperplanes H_1, H_2 and H_3 of $\mathcal{H}$.

Lemma 3.1 *Let $\mathcal{S}$ be an element of $\mathcal{E}_n$, $n \geq 1$, and let L be a line of $\mathcal{S}$. Then the set $\mathcal{H} = \{H_x \,|\, x \in L\}$ is a pencil of hyperplanes of $\mathcal{S}$.*

Proof. Let y be a point of $\mathcal{S}$. If y has distance at most $n-1$ from each point of L, then y is contained in every hyperplane of $\mathcal{H}$. If y has distance n from a point of L, then by property (E_4), y is contained in precisely one hyperplane of $\mathcal{H}$. This proves the lemma. ∎

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be an element of $\mathcal{E}_n$, $n \geq 1$, and let $e : \mathcal{S} \to \Sigma$ be a full embedding of $\mathcal{S}$. For every point x of $\mathcal{S}$, the singular hyperplane H_x is a maximal subspace of $\mathcal{S}$ and hence $\langle e(H_x) \rangle$ is either Σ or a hyperplane of Σ. We call the embedding e *polarized* if $\langle e(H_x) \rangle$ is a hyperplane of Σ for every point x of $\mathcal{S}$. If e is a polarized embedding, then since H_x is a maximal subspace, $\langle e(H_x) \rangle \cap e(\mathcal{P}) = e(H_x)$ for every point x of $\mathcal{S}$.

Proposition 3.2 *Let $\mathcal{S} \in \mathcal{E}_n$, $n \geq 1$. Let $e_1 : \mathcal{S} \to \Sigma_1$ and $e_2 : \mathcal{S} \to \Sigma_2$ be two full embeddings of $\mathcal{S}$ such that $e_1 \geq e_2$. If e_2 is polarized, then also e_1 is polarized.*

Proof. We know that $e_2 \cong e_1/U$ for some subspace U of Σ_1. Let x be an arbitrary point of $\mathcal{S}$. Since e_2 is polarized, $\langle e_2(H_x) \rangle$ is a hyperplane of Σ_1/U, i.e. a hyperplane Π of Σ_1 through U. Obviously, $\langle e_1(H_x) \rangle = \Pi$. So, also e_1 is polarized. ∎

Corollary 3.3 *Let $\mathcal{S} \in \mathcal{E}_n$, $n \geq 1$, and let e be a full polarized embedding of $\mathcal{S}$. Then also $\widetilde{e}$ is a full polarized embedding.*

Proposition 3.4 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be an element of $\mathcal{E}_n$, $n \geq 1$, and let $e : \mathcal{S} \to \Sigma$ be a full polarized embedding of $\mathcal{S}$. Put $R_e := \bigcap_{p \in \mathcal{P}} \langle e(H_p) \rangle$. Then R_e satisfies the conditions (C1) and (C2) of Section 2 and the embedding $\bar{e} := e/R_e$ is polarized.*

Proof. (i) We show that R_e satisfies property (C1). Let x be an arbitrary point of $\mathcal{S}$. By property (E_3), there exists a point p at distance n from x. Since $x \notin H_p$ and $e(\mathcal{P}) \cap \langle e(H_p) \rangle = e(H_p)$, $e(x) \notin \langle e(H_p) \rangle$. Hence, $e(x) \notin R_e$.

(ii) We show that R_e satisfies property (C2). Let x_1 and x_2 be two distinct points of $\mathcal{S}$. By property (E_3), there exists a point x at distance n from x_1 such that x_2 is on a geodesic from x_1 to x. Then $e(x_2) \in \langle e(H_x) \rangle$ and hence $\langle R_e, e(x_2) \rangle \subseteq \langle e(H_x) \rangle$. On the other hand, since $x_1 \notin H_x$, $e(x_1) \notin \langle e(H_x) \rangle$. Hence, $\langle R_e, e(x_2) \rangle \neq \langle R_e, e(x_1) \rangle$.

(iii) We show that $\bar{e}$ is polarized. Let x denote an arbitrary point of $\mathcal{S}$. Since e is polarized, $\langle e(H_x) \rangle$ is a hyperplane of Σ. Since $R_e \subseteq \langle e(H_x) \rangle$, $\langle e(H_x) \rangle$ determines a hyperplane of Σ/R_e which contains all points of $\bar{e}(H_x)$. This proves that $\bar{e}$ is polarized. ■

Suppose that $e : \mathcal{S} \to \Sigma$ is a full polarized embedding of an element $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ of $\mathcal{E}_n$, $n \geq 1$. Then $e^*(x) := \langle e(H_x) \rangle$ is a hyperplane of Σ for every point x of $\mathcal{S}$. Suppose that for every line L of $\mathcal{S}$, $\{e^*(x) \,|\, x \in L\}$ is a line in the dual Σ^* of Σ. Then e^* defines an embedding of $\mathcal{S}$ in a subspace $\Sigma^{(*)}$ of Σ^*, which we call the *dual embedding* of e. Notice that if $\dim(\Sigma)$ is finite, then there exists a natural bijective correspondence between the subspaces of Σ and those of Σ^*. In this case, the subspace R_e (as defined in Proposition 3.4) is the subspace of Σ corresponding with the subspace $\Sigma^{(*)}$ of Σ^*.

Proposition 3.5 *The dual embedding e^* (if it exists) is polarized.*

Proof. Let x be an arbitrary point of $\mathcal{S}$ and let H_x be the singular hyperplane with deepest point x. Since H_x is a maximal subspace, the subspace of $\Sigma^{(*)}$ generated by $e^*(H_x)$ is either $\Sigma^{(*)}$ or a hyperplane of $\Sigma^{(*)}$. But since $e(x) \in e^*(y)$ for every $y \in H_x$ and $e(x) \notin e^*(y)$ for every point $y \notin H_x$, the subspace of $\Sigma^{(*)}$ generated by $e^*(H_x)$ must be a hyperplane of $\Sigma^{(*)}$. Hence, e^* is polarized. ■

Proposition 3.6 *Let $\mathcal{S} \in \mathcal{E}_n$, $n \geq 1$, and let e_1 and e_2 be two full polarized embeddings of $\mathcal{S}$ such that $e_1 \geq e_2$. If one of e_1, e_2 has a dual embedding, then both e_1 and e_2 have dual embeddings and $e_1^* \cong e_2^*$.*

Proof. Let U denote the subspace of Σ_1 satisfying $(C1)$, $(C2)$ and $e_2 \cong e_1/U$. Since e_2 is polarized, also e_1/U is polarized. Let x denote an arbitrary point of $\mathcal{S}$ and let H_x denote the singular hyperplane with deepest point x. Since e_1/U is polarized, $\langle e_1/U(H_x)\rangle = \langle \bigcup_{y \in H_x} \langle U, e_1(y)\rangle\rangle = \langle U, e_1(H_x)\rangle$ is a hyperplane of Σ_1. Now, $\langle e_1(H_x)\rangle$ is a hyperplane of Σ_1 and it follows that $\langle e_1/U(H_x)\rangle = \langle e_1(H_x)\rangle/U$. The proposition now readily follows. ∎

Proposition 3.7 *Let $\mathcal{S} \in \mathcal{E}_n$, $n \geq 1$, and let $e : \mathcal{S} \to \Sigma$ be a full polarized embedding with $\dim(\Sigma) < \infty$ which has a dual embedding e^*. Then also e^* has a dual embedding and $e^{**} \cong \bar{e}$, with $\bar{e}$ the full polarized embedding of $\mathcal{S}$ as defined in Proposition 3.4.*

Proof. Let $e^* : \mathcal{S} \to \Sigma^{(*)}$ denote the dual embedding of e. Since $\dim(\Sigma) < \infty$, we may identify the subspaces of Σ with those of Σ^*. Recall that R_e (as defined in Proposition 3.4) is the subspace of Σ corresponding with $\Sigma^{(*)}$. Define $e^{**}(x) := \bigcap_{y \in H_x} e^*(y)$ for every point x of $\mathcal{S}$. By the proof of Proposition 3.5, $e^{**}(x) = \langle R_e, e(x)\rangle$. If L is a line of $\mathcal{S}$, then $\{\langle R_e, e(x)\rangle \,|\, x \in L\}$ is a line of the dual of $\Sigma^{(*)}$. Hence, e^* has e^{**} as dual embedding. From the above treatment it is also clear that $e^{**} \cong \bar{e}$. ∎

Proposition 3.8 *Let $\mathcal{S} \in \mathcal{E}_n$, $n \geq 1$, and let $e_1 : \mathcal{S} \to \Sigma_1$ and $e_2 : \mathcal{S} \to \Sigma_2$ be two full polarized embeddings of $\mathcal{S}$ with $\dim(\Sigma_1), \dim(\Sigma_2) < \infty$ having respective dual embeddings e_1^* and e_2^*. Then $e_1^* \cong e_2^*$ if and only if $\widetilde{e_1} \cong \widetilde{e_2}$.*

Proof. If $\widetilde{e_1} \cong \widetilde{e_2}$, then $e_1^* \cong \widetilde{e_1}^* \cong \widetilde{e_2}^* \cong e_2^*$ by Proposition 3.6. Conversely, suppose that $e_1^* \cong e_2^*$. Then $\bar{e}_1 \cong \bar{e}_2$ by Proposition 3.7 and hence $\widetilde{e_1} \cong \widetilde{\bar{e}_1} \cong \widetilde{\bar{e}_2} \cong \widetilde{e_2}$. ∎

Consider now the following question for a certain geometry $\mathcal{S} \in \mathcal{E}_n$, $n \geq 2$:

$(*)$ Has every full polarized embedding of $\mathcal{S}$ a dual embedding?

By Cardinali, De Bruyn and Pasini [6], we know that the answer to the above question is affirmative for thick dual polar spaces. Our Main Theorem, which we will prove in the following section, generalizes this result to arbitrary dense near polygons satisfying property (P_{de}). From Shult [13, Theorem 7.1], it readily follows (see Proposition 3.9 below) that the answer to question $(*)$ is also affirmative for all geometries $\mathcal{S} \in \mathcal{E}$ whose symplecta have polar rank at least 3. Notice that for the nondegenerate polar spaces, this already follows from the work of Veldkamp [17].

Proposition 3.9 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be an element of $\mathcal{E}_n$, $n \geq 2$, whose symplecta have polar rank at least 3. Then every full polarized embedding $e : \mathcal{S} \to \Sigma$ has a dual embedding.*

Proof. By Shult [13, Theorem 7.1], $\mathcal{S}$ admits Veldkamp lines. This means the following:

(i) If A and B are two distinct hyperplanes of $\mathcal{S}$, then A is not contained in B.

(ii) If A, B and C are three distinct hyperplanes such that $A \cap B \subseteq C$, then $A \cap B = A \cap C = B \cap C$.

We will make use of the following property.

Property. *Let $\mathcal{H}$ be a pencil of hyperplanes of $\mathcal{S}$ and let H_1 and H_2 be two distinct elements of $\mathcal{H}$. Then $\mathcal{H}$ coincides with the set $\mathcal{H}'$ of all hyperplanes through $H_1 \cap H_2$.*

PROOF. Obviously, $\mathcal{H} \subseteq \mathcal{H}'$. Suppose that there exists a hyperplane $H' \in \mathcal{H}' \setminus \mathcal{H}$. Since H' is a maximal subspace and $H_1 \cap H_2$ is not a maximal subspace, there exists a point $y \in H' \setminus (H_1 \cap H_2)$. Let H denote the unique element of $\mathcal{H}$ through y and let $i \in \{1, 2\}$ such that $H_i \neq H$. Then the triple $\{H, H', H_i\}$ contradicts property (ii) in the definition of Veldkamp line. (qed)

Now, let L be a line of $\mathcal{S}$ through two distinct collinear points x_1 and x_2, and let $\Pi_i := \langle e(H_{x_i}) \rangle$, $i \in \{1, 2\}$. Consider the following two pencils of hyperplanes of $\mathcal{S}$:

- $\mathcal{H}_1 := \{H_x \,|\, x \in L\}$ (see Lemma 3.1);
- $\mathcal{H}_2 := \{e^{-1}(e(\mathcal{P}) \cap \Pi) \,|\, \Pi$ is a hyperplane of Σ through $\Pi_1 \cap \Pi_2\}$.

The hyperplanes H_{x_1} and H_{x_2} are contained in $\mathcal{H}_1$ and $\mathcal{H}_2$. Hence $\mathcal{H}_1 = \mathcal{H}_2$ by the previous property. It now readily follows that e has a dual embedding. ■

The answer to question $(*)$ is also affirmative for geometries of $\mathcal{E}$ containing only lines with three points, as we will prove now. Notice first that if H_1 and H_2 are two distinct hyperplanes of a partial linear space $\mathcal{S}$ with three points on each line, then the complement $\overline{H_1 \Delta H_2}$ of the symmetric difference of H_1 and H_2 is again a hyperplane.

Lemma 3.10 *Let $\mathcal{S}$ be an element of $\mathcal{E}_n$, $n \geq 1$, with three points on every line and let $L = \{x_1, x_2, x_3\}$ be a line of $\mathcal{S}$. Then the singular hyperplane H_{x_3} is equal to the hyperplane $\overline{H_{x_1} \Delta H_{x_2}}$.*

Proof. This is an immediate corollary of Lemma 3.1. ■

Proposition 3.11 *Let $\mathcal{S}$ be an element of $\mathcal{E}_n$, $n \geq 1$, and suppose that every line of $\mathcal{S}$ contains precisely three points. Then* (*i*) *there exists at least one full polarized embedding of* $\mathcal{S}$;

(*ii*) *every full polarized embedding of* $\mathcal{S}$ *has a dual embedding.*

Proof. (i) Let W denote the set of all hyperplanes of $\mathcal{S}$ union $\{\mathcal{S}\}$. Then W has the structure of a $\mathbb{F}_2$-vectorspace if we take the following addition and scalar multiplication ($w_1, w_2 \in W$): $0 \cdot w_1 = \mathcal{S}$, $1 \cdot w_1 = w_1$, $w_1 + w_2 := \overline{w_1 \Delta w_2}$. For every point x of $\mathcal{S}$, let H_x denote the singular hyperplane with deepest point x. Let V denote the subspace of W generated by all singular hyperplanes. For every point x of $\mathcal{S}$, the set of all elements of V containing the point x obviously determines a hyperplane V_x of V. By Lemma 3.10, the map $p \to H_p$ defines a full projective embedding e of $\mathcal{S}$ in $\mathrm{PG}(V)$. Notice that this map is injective by (E_3). The embedding e is polarized since $e(H_p) \subseteq V_p$ for every point p of $\mathcal{S}$.

(ii) Let $e : \mathcal{S} \to \Sigma$ be a full polarized embedding of $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$. Put $e^*(x) := \langle e(H_x) \rangle$ for every point x of $\mathcal{S}$. Let $L = \{x_1, x_2, x_3\}$ be an arbitrary line of $\mathcal{S}$ and let Π be the unique hyperplane of Σ through $e^*(x_1) \cap e^*(x_2)$ different form $e^*(x_1)$ and $e^*(x_2)$. Then $e^{-1}(\Pi \cap e(\mathcal{P}))$ coincides with $\overline{H_{x_1} \Delta H_{x_2}} = H_{x_3}$. Hence, $e^*(x_1)$, $e^*(x_2)$ and $e^*(x_3)$ form a line of Σ^*. This proves that e^* is an embedding. ■

Remarks. By Ronan [11] and Proposition 3.11, every element $\mathcal{S}$ of $\mathcal{E}_n$, $n \geq 1$, with three points on each line has an absolutely universal embedding. Suppose now that the absolutely universal embedding space is finite-dimensional. By Proposition 3.8, it then follows that all dual embeddings are isomorphic to a certain embedding e_m. This embedding e_m is called the *minimal full polarized embedding* of $\mathcal{S}$. If e is a full polarized embedding of $\mathcal{S}$, then $e \geq e_m$ by Proposition 3.7. The minimal full polarized embedding of a slim dense near polygon is also called the *near polygon embedding*, see Brouwer and Shpectorov [3].

4 Completion of the proof of the Main Theorem

Lemma 4.1 *Let L be a line of a dense near $2n$-gon $\mathcal{S}$ ($n \geq 1$), let x be a point of L and let V denote the set of all convex subspaces of diameter $n-1$ through x not containing L. Let $\Gamma_{x,L}$ be the graph with vertex set V, with two vertices adjacent whenever they intersect in a convex subspace of diameter $n-2$. Then the graph $\Gamma_{x,L}$ is connected.*

Proof. Let A_1 and A_2 be two arbitrary vertices of $\Gamma_{x,L}$. We will prove by downwards induction on $\mathrm{diam}(A_1 \cap A_2)$ that A_1 and A_2 are connected by a path. Obviously, this holds if $\mathrm{diam}(A_1 \cap A_2) \geq n-2$. So, suppose

$\mathrm{diam}(A_1 \cap A_2) = n - \delta$ with $\delta \geq 3$. Let L_1 denote a line of A_1 through x not contained in $A_1 \cap A_2$. [Such a line exists since a convex subspace F of a dense near polygon is completely determined by the neighbourhood $\Gamma_1(u) \cap F$ of one its points $u \in F$, see e.g. Theorem 2.14 of [8]. Notice also that $A_1 \cap A_2$ is properly contained in A_1.] Then $\langle A_1 \cap A_2, L_1, L \rangle$ is a convex subspace of diameter $n - \delta + 2 \leq n - 1$ different from A_2. Hence, there exists a line L_2 in A_2 through x not contained in $\langle A_1 \cap A_2, L_1, L \rangle$. [Similar explanation as above. Notice also that $\langle A_1 \cap A_2, L_1, L \rangle \cap A_2$ is properly contained in A_2.] Now, the convex subspace $\langle A_1 \cap A_2, L_1, L_2 \rangle$ has diameter $n - \delta + 2 \leq n - 1$ not containing the line L. We will now show that there exists a convex subspace A_3 of diameter $n - 1$ through $\langle A_1 \cap A_2, L_1, L_2 \rangle$ not containing the line L. Take a point $u \in L \setminus \{x\}$ and a point v of $\langle A_1 \cap A_2, L_1, L_2 \rangle$ at maximal distance $n - \delta + 2$ from x. Then there exists a point w at maximal distance n from u such that v is on a geodesic from u to w. Since $\mathrm{d}(u, v) = \mathrm{d}(u, x) + \mathrm{d}(x, v) = 1 + \mathrm{d}(x, v)$, we necessarily have $\mathrm{d}(x, w) = \mathrm{d}(x, v) + \mathrm{d}(v, w) = n - 1$. Clearly, $A_3 := \langle x, w \rangle$ has diameter $n - 1$, contains $\langle A_1 \cap A_2, L_1, L_2 \rangle = \langle x, v \rangle$ (since v is on a geodesic from x to w), but not the line L (since $\mathrm{d}(w, u) = n$). Now, by the induction hypothesis, A_3 and A_i ($i \in \{1, 2\}$) are connected by a path. Hence, also A_1 and A_2 are connected by a path. ∎

Lemma 4.2 *Let a and b be two distinct collinear points of a dense near $2n$-gon $\mathcal{S}$ ($n \geq 1$) satisfying property (P_{de}). If H is a hyperplane of $\mathcal{S}$ such that $H \cap (H_a \cup H_b) = H_a \cap H_b$, then $H = H_c$ for some point c on the line $L = ab$ different from a and b.*

Proof. Notice first that:

(i) Every hyperplane of $\mathcal{S}$ is a maximal subspace. In particular, this property holds for the singular hyperplanes (recall [13, Lemma 6.1]).

(ii) $H_a \cap H_b$ consists of those points of $\mathcal{S}$ at distance at most $n-2$ from L.

(iii) $H_a \cap H_b$ is not a maximal subspace, since it is contained in the two distinct maximal subspaces H_a and H_b.

By (i) and (iii), there exists a point $u \in H \setminus (H_a \cap H_b)$ and by (ii), $\mathrm{d}(u, L) = n - 1$. Let c denote the unique point of L nearest to u. Since $u \notin H_a \cup H_b$, $a \neq c \neq b$.

Step 1: $\langle u, c \rangle \subseteq H$.
Let v denote an arbitrary point of $\langle u, c \rangle$. If $\mathrm{d}(c, v) \leq n - 2$, then $v \in H$ since $v \in H_a \cap H_b$. Suppose now that $\mathrm{d}(c, v) = n - 1$. Since the hyperplane $\Gamma_{\leq n-2}(c) \cap \langle u, c \rangle$ of $\langle u, c \rangle$ is a maximal subspace, $\Gamma_{n-1}(c) \cap \langle u, c \rangle$

is connected. So, there exists a path $w_0, w_1, w_2, \ldots, w_k$ in $\Gamma_{n-1}(c) \cap \langle u, c\rangle$ connecting the points $u = w_0$ and $v = w_k$. If $w_i \in H$ for a certain $i \in \{0, \ldots, k-1\}$, then also $w_{i+1} \in H$ since $w_i \in H$ and $w_i w_{i+1} \cap \Gamma_{n-2}(c) \subseteq H$. Since $w_0 \in H$, it follows that also $v = w_k \in H$.

Step 2: *If A_1 and A_2 are two adjacent vertices of $\Gamma_{c,L}$ such that $A_1 \subseteq H$, then also $A_2 \subseteq H$.*
Put $A_3 := \langle A_1 \cap A_2, L\rangle$. Since $\mathcal{S}$ satisfies property (P_{de}), the graph $\Gamma(c, A_1 \cap A_2, A_3)$ is connected. So, it suffices to prove the claim in the case that A_1 and A_2 are adjacent vertices of $\Gamma(c, A_1 \cap A_2, A_3)$. There exists then a quad Q such that: (i) $Q \cap (A_1 \cap A_2)$ is a point y at distance $n-2$ from c; (ii) for every $i \in \{1, 2, 3\}$, $L_i := Q \cap A_i$ is a line. Notice that the lines L_1 and L_3 of Q are contained in H. There are two possibilities:

(1) Q is subquadrangular or deep with respect to H;

(2) Q is singular with respect to H.

Suppose case (1) occurs. Put $a^* := \pi_{L_3}(a)$ and let x denote an arbitrary point of $(\Gamma_1(a^*) \cap Q) \setminus L_3$ contained in H. Since $\mathrm{d}(x, A_3) = \mathrm{d}(x, a^*) = 1$, $\Gamma_{n-1}(x) \cap L = \Gamma_{n-2}(a^*) \cap L = \{a\}$. So, x is a point of $H \cap (H_a \cup H_b)$ not contained in $H_a \cap H_b$, a contradiction.

Hence, case (2) occurs. Then Q is singular with deep point $L_1 \cap L_3$. It follows that $L_2 \subseteq H$. Hence, $\Gamma_{n-1}(c) \cap A_2$ contains a point of H. By Step 1, it then follows that $A_2 \subseteq H$.

Step 3: $H = H_c$.
Since H_c is a maximal subspace, it suffices to show that $H_c \subseteq H$ or that any convex subspace A of diameter $n-1$ through c is contained in H. If $L \subseteq A$, then $A \subseteq H_a \cap H_b \subseteq H$. Hence, we may suppose that $L \not\subseteq A$, or that A is a vertex of $\Gamma_{c,L}$. By Step 2, Lemma 4.1 and the fact that $\langle u, c\rangle \subseteq H$, it follows that $A \subseteq H$. This proves the statement. ∎

We can now complete the proof of the Main Theorem. Let L be an arbitrary line of a dense near polygon $\mathcal{S}$ which satisfies property (P_{de}) and let $e : \mathcal{S} \to \Sigma$ be a full polarized embedding of $\mathcal{S}$. We must show that $\{e^*(x) \,|\, x \in L\}$ is a line of the dual Σ^* of Σ. Let x_1 and x_2 denote two distinct points of L. If Π is a hyperplane of Σ through $e^*(x_1) \cap e^*(x_2)$, then by Lemma 4.2, $e^{-1}(\Pi \cap e(\mathcal{P})) = H_c$ for some point c of L different from x_1 and x_2. Conversely, take an arbitrary point c' on L different from x_1 and x_2 and let d be an arbitrary point of $\Gamma_{n-1}(L) \cap \Gamma_{n-1}(c')$. Then $e(d)$ is not contained in $e^*(x_1) \cap e^*(x_2)$, since $(e^*(x_1) \cap e^*(x_2)) \cap e(\mathcal{P}) = (e^*(x_1) \cap e(\mathcal{P})) \cap (e^*(x_2) \cap e(\mathcal{P})) = e(H_{x_1}) \cap e(H_{x_2}) = e(H_{x_1} \cap H_{x_2})$. So, $\Pi' := \langle e(d), e^*(x_1) \cap e^*(x_2)\rangle$ is a hyperplane of Σ. As before, $e^{-1}(\Pi' \cap e(\mathcal{P}))$ is a singular hyperplane of $\mathcal{S}$ with deepest point on L. It follows that $e^{-1}(\Pi' \cap e(\mathcal{P})) = H_{c'}$ since $d \in e^{-1}(\Pi' \cap e(\mathcal{P}))$. This proves that $\{e^*(x) \,|\, x \in L\}$ is a line of Σ^*.

5 Dense near polygons satisfying property (P_{de})

Every known dense near polygon either belongs to a certain list of near polygons or is obtained by applying the so-called direct-product and glueing constructions to near polygons of that list. The list includes the dual polar spaces, the near hexagons $\mathbb{E}_1$, $\mathbb{E}_2$, $\mathbb{E}_3$ and the infinite classes $\mathbb{G}_n$, $\mathbb{H}_n$, $\mathbb{I}_n$ $(n \geq 3)$. For each of these near polygons we will now determine whether they satisfy property (P_{de}) or not. From Propositions 5.1, 5.2, 5.3, 5.4, 5.5, 5.6, 5.7, 5.8 and 5.9 below, it follows that every known dense near $2n$-gon, $n \geq 2$, satisfies property (P_{de}), except for the ones containing an $\mathbb{E}_1$-hex or an $\mathbb{E}_2$-hex. From the Main Theorem, it then readily follows that every full polarized embedding of a known dense near polygon has a dual embedding.

5.1 Dual polar spaces

With every nondegenerate polar space Π of rank $n \geq 2$, there is associated a point-line geometry Δ, whose points are the maximal singular subspaces of Π, whose lines are the next-to-maximal singular subspaces of Π and whose incidence relation is reverse containment. The geometry Δ is called a *dual polar space*. The dual polar spaces of rank 2 are precisely the nondegenerate generalized quadrangles. Every convex subspace of a dual polar space Δ is classical in Δ.

Proposition 5.1 *Let Δ be a dual polar space of rank at least 2 with at least 3 points on every line. Then Δ satisfies property (P_{de}). In particular, every nondegenerate generalized quadrangle with at least 3 points per line satisfies property (P_{de}).*

Proof. We will show that Δ satisfies property (P'_{de}). So, let B be an arbitrary convex subspace of diameter $n-2$, let x denote an arbitrary point of B and let A_1, A_2 and A_3 be three convex subspaces of diameter $n-1$ through B. Take a quad Q through x intersecting B only in the point x. Since A_i, $i \in \{1, 2, 3\}$, is big in Δ, $A \cap B_i$ is a line (recall Theorem 2.32 of [8]). This proves that Δ satisfies property (P'_{de}) and hence also property (P_{de}). ∎

5.2 The near hexagon $\mathbb{E}_1$

Let C be the *extended ternary Golay code*, i.e. the 6-dimensional subspace of $\mathbb{F}_3^{12}$ generated by the rows of the following matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 1 & 0 & -1 \end{bmatrix}.$$

With C, there is associated a near hexagon $\mathbb{E}_1$, see Shult and Yanushka [14] or De Bruyn [8, Section 6.5]. The points of $\mathbb{E}_1$ are the cosets of C in $\mathbb{F}_3^{12}$ and two cosets are collinear whenever they contain vectors which differ in only one position.

Proposition 5.2 *The near hexagon $\mathbb{E}_1$ does not satisfy property (P_{de}).*

Proof. The near hexagon $\mathbb{E}_1$ cannot satisfy property (P_{de}) since every quad is a grid. ■

5.3 The near hexagon $\mathbb{E}_2$

Let $S(5,8,24)$ be the unique 5-(24,8,1)-design. [Such a design is an incidence structure of points and blocks satisfying (i) there are precisely **24** points, (ii) each block contains precisely **8** points, (iii) every **5** distinct points are contained in precisely **1** block.] With the design $S(5,8,24)$, there is associated a near hexagon $\mathbb{E}_2$, see Shult and Yanushka [14] or [8, Section 6.6]. The points of $\mathbb{E}_2$ are the blocks of $S(5,8,24)$ and the lines are the triples of mutually disjoint blocks (natural incidence).

Proposition 5.3 *The near hexagon $\mathbb{E}_2$ does not satisfy property (P_{de}).*

Proof. Let x be a point of $\mathbb{E}_2$, let B be a line through x and let A be a quad through B. We will show that the graph $\Gamma(x,B,A)$ has three connected components. Put $B=\{x,x_1,x_2\}$

Claim. *Suppose A_1 and A_2 are two quads through B such that A_1, A_2 and A are mutually different. If there exists a quad Q_1 through x_1 satisfying (i) $Q_1 \cap B = \{x_1\}$ and (ii) $Q_1 \cap A$, $Q_1 \cap A_1$ and $Q_1 \cap A_2$ are lines, then there exists a quad Q_2 through x_2 satisfying (i) $Q_2 \cap B = \{x_2\}$ and (ii) $Q_2 \cap A$, $Q_2 \cap A_1$ and $Q_2 \cap A_2$ are lines.*

Let L_i, $i \in \{1,2\}$, be a line of A_i through x_2 different from B. Then the quad $Q_2 := \langle L_1, L_2 \rangle$ does not contain B and hence is disjoint from Q_1. Now, disjoint quads of $\mathbb{E}_2$ can only have one kind of mutual position, see Section 4 of [2]. It follows that $Q_2 \cap \Gamma_1(Q_1)$ consists of a point of Q_2 together with all its neighbours. Since $L_1, L_2 \subseteq Q_2 \cap \Gamma_1(Q_1)$, $Q_2 \cap \Gamma_1(Q_1) = L_1 \cup L_2 \cup L_3$, where L_3 is the third line of Q_2 through x_2. Now, let L_3' denote the unique line of Q_1 such that every point of L_3' is collinear with a unique point of L_3. Then $L_3' \subseteq A$. Every point of $L_3 \setminus \{x_2\}$ is collinear with the point x_2 and with a point of L_3'. It follows that $L_3 \subseteq A$. This proves the claim.

Now, consider the local space $\mathcal{L} = \mathcal{L}(\mathcal{S}, x_2)$ which is isomorphic to PG$(3, 2)$, see [2, Section 4.3]. For every convex subspace F through x_2, let $\widetilde{F}$ denote the corresponding subspace of $\mathcal{L}$. Then $\widetilde{B}$ is a point of $\mathcal{L}$ and $\widetilde{A}$ is a line of $\mathcal{L}$ through $\widetilde{B}$. Now, suppose that A_1 and A_2 are two adjacent vertices of $\Gamma(x, B, A)$. Then by the previous claim, there exists a line $\widetilde{Q}$ in $\mathcal{L}$ which intersects the lines $\widetilde{A}$, $\widetilde{A_1}$ and $\widetilde{A_2}$ in points different from $\widetilde{B}$. This implies that the lines $\widetilde{A_1}$ and $\widetilde{A_2}$ are contained in the same plane of $\mathcal{L}$ through $\widetilde{A}$. It is now obvious that the graph $\Gamma(x, B, A)$ has three connected components. ■

5.4 The near hexagon $\mathbb{E}_3$

Consider in PG$(6, 3)$ a nonsingular parabolic quadric $Q(6, 3)$ and a non-tangent hyperplane π intersecting $Q(6, 3)$ in a nonsingular elliptic quadric $Q^-(5, 3)$. There is a polarity associated with $Q(6, 3)$ and we call two points *orthogonal* when one of them is contained in the polar hyperplane of the other. Let N denote the set of 126 internal points of $Q(6, 3)$ which are contained in π, i.e. the set of all 126 points in π for which the polar hyperplane intersects $Q(6, 3)$ in a nonsingular elliptic quadric. The following near hexagon $\mathbb{E}_3$ can now be constructed, see [4, Section (n)]. The points of $\mathbb{E}_3$ are the 6-tuples of mutually orthogonal points of N, the lines of $\mathbb{E}_3$ are the pairs of mutually orthogonal points of N and incidence is reverse containment. Group-theoretical constructions for $\mathbb{E}_3$ can be found in Aschbacher [1], Kantor [9] and Ronan and Smith [12]. Every local space of $\mathbb{E}_3$ is isomorphic to $\overline{W(2)}$, the linear space derived from $W(2)$ by adding its ovoids as extra lines.

Proposition 5.4 *The near hexagon $\mathbb{E}_3$ satisfies property (P_{de}).*

Proof. We will show that $\mathbb{E}_3$ satisfies property (P'_{de}). So, let x denote an arbitrary point of $\mathbb{E}_3$, let L denote an arbitrary line through x and let Q_1, Q_2, Q_3 be three distinct quads through L. Let Q be one of the four $Q(5, 2)$-quads through x not containing the line L. Then Q is big in $\mathbb{E}_3$ and hence $Q \cap Q_i$ is a line for every $i \in \{1, 2, 3\}$. This proves that $\mathbb{E}_3$ satisfies property (P'_{de}) and hence also property (P_{de}). ■

5.5 The near polygon $\mathbb{I}_n$, $n \geq 3$

Let $Q(2n, 2)$, $n \geq 2$, be a nonsingular parabolic quadric of $\mathrm{PG}(2n, 2)$ and let Π be a hyperplane of $\mathrm{PG}(2n, 2)$ intersecting $Q(2n, 2)$ in a nonsingular hyperbolic quadric $Q^+(2n-1, 2)$. The generators of $Q(2n, 2)$ are the points of a dual polar space $DQ(2n, 2)$. The generators of $Q(2n, 2)$ not contained in $Q^+(2n-1, 2)$ form a subspace of $DQ(2n, 2)$. The points and lines contained in this subspace define a dense near $2n$-gon which we will denote by $\mathbb{I}_n$. Let V denote the set of all subspaces of $Q(2n, 2)$ with exception of the $(n-2)$- and $(n-1)$-dimensional subspaces contained in Π. There exists a bijective correspondence between the nonempty convex subspaces of $\mathbb{I}_n$ and the elements of V: if α is a subspace of V, then the generators through α define a convex subspace of $\mathbb{I}_n$. For more details on the above-mentioned facts, we refer to [8, Chapter 6].

Proposition 5.5 *The near $2n$-gon $\mathbb{I}_n$, $n \geq 3$, satisfies property (P_{de}).*

Proof. We will show that $\mathbb{I}_n$ satisfies property (P'_{de}). Let x denote an arbitrary point of $\mathbb{I}_n$, let B denote an arbitrary convex subspace of diameter $n-2$ through x and let A_1, A_2 and A_3 be three distinct convex subspaces of diameter $n-1$ through B. For every convex subspace F of $\mathbb{I}_n$, let $\widetilde{F}$ denote the corresponding subspace of $Q(2n, 2)$. The point x corresponds with a generator $\widetilde{x}$ of $Q(2n, 2)$. The subspace $\widetilde{B}$ is a line of $\widetilde{x}$ and $\widetilde{A_1}$, $\widetilde{A_2}$ and $\widetilde{A_3}$ are three points of $\widetilde{B}$. Now, there always exists a subspace $\widetilde{Q}$ of dimension $n-3$ in $\widetilde{x}$ which is disjoint from $\widetilde{B}$ and which is not contained in the $(n-2)$-dimensional subspace $\widetilde{x} \cap \Pi$. The quad Q corresponding with $\widetilde{Q}$ intersects B in the point x and A_1, A_2 and A_3 in the respective lines L_1, L_2 and L_3 with $\widetilde{L_i} = \langle \widetilde{Q}, \widetilde{A_i} \rangle$, $i \in \{1, 2, 3\}$. This proves that $\mathbb{I}_n$ satisfies property (P'_{de}) and hence also property (P_{de}). ∎

5.6 The near polygon $\mathbb{H}_n$, $n \geq 3$

Let X be a set of size $2n+2$, $n \geq 2$. The following near $2n$-gon can then be constructed. The points of $\mathbb{H}_n$ are the partitions of X in $n+1$ subsets of size 2 and the lines are the partitions of X in $n-1$ subsets of size 2 and 1 subset of size 4. A point is incident with a line if and only if the partition corresponding with the point is a refinement of the partition corresponding with the line. There exists a bijective correspondence between the convex sub-2δ-gons of $\mathbb{H}_n$ and the partitions of X in $n+1-\delta$ subsets of even size. If P is such a partition, then the points of $\mathbb{H}_n$ which are a refinement of P define a convex sub-2δ-gon of $\mathbb{H}_n$. For more details on the above-mentioned facts, we refer to [8, Chapter 6].

Proposition 5.6 *The near $2n$-gon $\mathbb{H}_n$, $n \geq 3$, satisfies property (P_{de}).*

Proof. We will prove that $\mathbb{H}_n$ satisfies property (P'_{de}). Let x denote an arbitrary point of $\mathbb{H}_n$, let B denote an arbitrary sub-$(2n-4)$-gon through x and let A_1, A_2 and A_3 denote (the) three distinct sub-$(2n-2)$-gons through B. Let $\{X_1, X_2, X_3\}$ denote the partition of X corresponding with B. The partition P_x corresponding with x is a refinement of $\{X_1, X_2, X_3\}$. Without loss of generality, we may suppose that the convex sub-$(2n-2)$-gon A_i, $i \in \{1, 2, 3\}$, corresponds with the partition $\{X_i, X_{i+1} \cup X_{i+2}\}$, where the indices are taken modulo 3. Let v_1, v_2, v_3 be three subsets of size 2 of P_x such that $v_1 \subseteq X_1$, $v_2 \subseteq X_2$ and $v_3 \subseteq X_3$. Let Q be the quad of $\mathbb{H}_n$ corresponding with the partition $P_Q := P_x \setminus \{v_1, v_2, v_3\} \cup \{v_1 \cup v_2 \cup v_3\}$. It is obvious that:

(i) P_x is the unique common refinement of P_Q and $\{X_1, X_2, X_3\}$ in subsets of even size. Hence, Q intersects B in the point x.

(ii) The partition $P_x \setminus \{v_{i+1}, v_{i+2}\} \cup \{v_{i+1} \cup v_{i+2}\}$ is a refinement of P_Q and $\{X_i, X_{i+1} \cup X_{i+2}\}$ $(i \in \{1, 2, 3\})$. Hence, Q and A_i intersect in a line.

This proves that $\mathbb{H}_n$ satisfies property (P'_{de}) and hence also property (P_{de}). ■

5.7 The near polygon $\mathbb{G}_n$, $n \geq 3$

Let $H(2n-1, 4)$, $n \geq 2$, denote a nonsingular hermitian variety in $\mathrm{PG}(2n-1, 4)$. Without loss of generality, we may suppose that $H(2n-1, 4)$ has equation $X_0^3 + X_1^3 + \cdots + X_{2n-1}^3 = 0$ with respect to a certain reference system. Let X denote the set of points of $\mathrm{PG}(2n-1, 4)$ with precisely two nonzero coordinates.

The generators of $H(2n-1, 4)$ are the points of a dual polar space $DH(2n-1, 4)$. The generators of $H(2n-1, 4)$ containing precisely n points of X form a subspace of $DH(2n-1, 4)$. The points and lines which are contained in this subspace define a near $2n$-gon which we will denote by $\mathbb{G}_n$. If $n \geq 3$, then $\mathrm{Aut}(\mathbb{G}_n)$ has two orbits on the set of lines. One orbit consists of those lines which are not contained in a $W(2)$-quad. We call these lines *special*. The other orbit consists of the so-called *ordinary lines*. Every point of $\mathbb{G}_n$ is contained in precisely n special lines and if $L_1, \ldots, L_k$ are $k \in \{2, \ldots, n\}$ such lines then $\langle L_1, \ldots, L_k \rangle \cong \mathbb{G}_k$. Since $\mathbb{G}_2 \cong Q(5, 2)$, every two intersecting special lines generate a $Q(5, 2)$-quad. For more details on the above-mentioned facts, we refer to [8, Chapter 6].

Proposition 5.7 *The near $2n$-gon $\mathbb{G}_n$, $n \geq 3$, satisfies property (P_{de}).*

Proof. We will show that $\mathbb{G}_n$ satisfies property (P'_{de}). Let x denote an arbitrary point of $\mathbb{G}_n$, let B denote a convex sub-$(2n-4)$-gon through x

and let A_1, A_2 and A_3 be three convex sub-$(2n-2)$-gons through B. Let $L_1, \ldots, L_n$ denote the n special lines through x. Without loss of generality, we may suppose that, for a certain $k \in \{0, \ldots, n\}$, the lines $L_1, \ldots, L_k$ are contained in B and the lines $L_{k+1}, \ldots, L_n$ are not contained in B. If $k \geq n-1$, then $\langle L_1, \ldots, L_k \rangle \cong \mathbb{G}_k$ has diameter at at least $n-1$ and is contained in B, a contradiction. Hence, $k \leq n-2$. Suppose that all quads $\langle L_i, L_j \rangle$, $i, j \in \{k+1, \ldots, n\}$ with $i \neq j$, meet B in a line. Then the sub-$(2n-2)$-gon $\langle B, L_n \rangle$ contains all special lines $L_1, \ldots, L_n$ and hence also $\mathbb{G}_n$, which is impossible. Hence, there exists a $Q(5,2)$-quad Q through x such that $Q \cap B = \{x\}$. Now, each $Q(5,2)$-quad is classical in $\mathbb{G}_n$ and it follows that Q intersects A_i, $i \in \{1,2,3\}$, in a line (recall [8, Theorem 2.32]). This proves that $\mathbb{G}_n$ satisfies property (P'_{de}) and hence also property (P_{de}). ∎

5.8 Product near polygons

Any two near polygons $\mathcal{A}_1 = (\mathcal{P}_1, \mathcal{L}_1, \mathrm{I}_1)$ and $\mathcal{A}_2 = (\mathcal{P}_2, \mathcal{L}_2, \mathrm{I}_2)$ of diameter at least 1 give rise to a so-called product near polygon $\mathcal{A}_1 \times \mathcal{A}_2$. The point set of $\mathcal{A}_1 \times \mathcal{A}_2$ is equal to $\mathcal{P}_1 \times \mathcal{P}_2$ and the line set is equal to $(\mathcal{P}_1 \times \mathcal{L}_2) \cup (\mathcal{L}_1 \times \mathcal{P}_2)$. (We assume here that $\mathcal{P}_1 \cap \mathcal{L}_1 = \emptyset$ and $\mathcal{P}_2 \cap \mathcal{L}_2 = \emptyset$.) The point (x, y) of $\mathcal{A}_1 \times \mathcal{A}_2$ is incident with the line $(z, L) \in \mathcal{P}_1 \times \mathcal{L}_2$ if and only if $x = z$ and $y \, \mathrm{I}_2 \, L$, the point (x, y) of $\mathcal{A}_1 \times \mathcal{A}_2$ is incident with the line $(M, u) \in \mathcal{L}_1 \times \mathcal{P}_2$ if and only if $x \, \mathrm{I}_1 \, M$ and $y = u$. The near polygon $\mathcal{A}_1 \times \mathcal{A}_2$ is called the *direct product* of $\mathcal{A}_1$ and $\mathcal{A}_2$ and its diameter is equal to the sum of the diameters of $\mathcal{A}_1$ and $\mathcal{A}_2$. There exists a partition T_i, $i \in \{1,2\}$, of $\mathcal{A}_1 \times \mathcal{A}_2$ in convex subpolygons isomorphic to $\mathcal{A}_i$ such that the following holds: (1) every subpolygon of T_1 intersects every subpolygon of T_2 in a point; (2) every line of $\mathcal{A}_1 \times \mathcal{A}_2$ is contained in a unique subpolygon of $T_1 \cup T_2$. Every subpolygon of $T_1 \cup T_2$ is classical in $\mathcal{A}_1 \times \mathcal{A}_2$ and if F and F' are two subpolygons of T_i, $i \in \{1,2\}$, then the map $F' \to F, x \mapsto \pi_F(x)$ defines an isomorphism between F' and F.

Let x denote an arbitrary point of $\mathcal{A}_1 \times \mathcal{A}_2$ and let $F_i(x)$, $i \in \{1,2\}$, denote the unique element of T_i through x. For every convex subspace U of $\mathcal{A}_1 \times \mathcal{A}_2$ through x, let U_i, $i \in \{1,2\}$, denote the convex subspace $F_i(x) \cap U$ of $F_i(x)$. It holds $\mathrm{diam}(U) = \mathrm{diam}(U_1) + \mathrm{diam}(U_2)$ and if $\mathrm{diam}(U_1), \mathrm{diam}(U_2) \geq 1$, then $U \cong U_1 \times U_2$.

If V_1 is a convex subspace of $F_1(x)$ through x and if V_2 is a convex subspace of $F_2(x)$ through x, then $U := \langle V_1, V_2 \rangle$ is a convex subspace of $\mathcal{A}_1 \times \mathcal{A}_2$ through x with $U_1 = V_1$ and $U_2 = V_2$.

Proposition 5.8 *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two dense near polygons.*

(i) *If $\mathrm{diam}(\mathcal{A}_1) = \mathrm{diam}(\mathcal{A}_2) = 1$, then $\mathcal{A}_1 \times \mathcal{A}_2$ satisfies property (P_{de}).*

(*ii*) *If* $\mathrm{diam}(\mathcal{A}_1) = 1$ *and* $\mathrm{diam}(\mathcal{A}_2) \geq 2$*, then* $\mathcal{A}_1 \times \mathcal{A}_2$ *satisfies property* (P_{de}) *if and only if* $\mathcal{A}_2$ *satisfies property* (P_{de})*.*

(*iii*) *If* $\mathrm{diam}(\mathcal{A}_1), \mathrm{diam}(\mathcal{A}_2) \geq 2$*, then* $\mathcal{A}_1 \times \mathcal{A}_2$ *satisfies property* (P_{de}) *if and only if* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *satisfy property* (P_{de})*.*

Proof. Put $\mathrm{diam}(\mathcal{A}_1 \times \mathcal{A}_2) = n$. Let x denote an arbitrary point of $\mathcal{A}_1 \times \mathcal{A}_2$, let B denote an arbitrary convex sub-$(2n-4)$-gon through x and suppose C, D and E are three convex sub-$(2n-2)$-gons through B. We will use the notations as introduced before this proposition. We have $\mathrm{diam}(B_1) + \mathrm{diam}(B_2) = \mathrm{diam}(B) = \mathrm{diam}(F_1(x)) + \mathrm{diam}(F_2(x)) - 2$. If $\mathrm{diam}(B_1) = \mathrm{diam}(F_1(x)) - 1$ and $\mathrm{diam}(B_2) = \mathrm{diam}(F_2(x)) - 1$, then B is contained in precisely two convex subpolygons of diameter $n-1$, namely $\langle F_1(x), B_2 \rangle$ and $\langle B_1, F_2(x) \rangle$, a contradiction. Hence, there exists an $i \in \{1,2\}$ such that $\mathrm{diam}(B_i) = \mathrm{diam}(F_i(x)) - 2$ and $B_{3-i} = F_{3-i}(x)$. We now show that D and E are adjacent vertices of $\Gamma(x, B, C)$ if and only of D_i and E_i are adjacent vertices of $\Gamma(x, B_i, C_i)$.

If Q is a quad meeting B in a point of $\Gamma_{n-2}(x)$ and C, D and E in lines, then Q is contained in a convex subspace of T_i and $\pi_{F_i(x)}(Q)$ is a quad meeting B_i in a point at distance $\mathrm{diam}(F_i(x)) - 2$ from x and C_i, D_i and E_i in lines. Conversely, suppose Q is a quad meeting B_i in a point at distance $\mathrm{diam}(F_i(x)) - 2$ from x and C_i, D_i and E_i in lines. Take a point y in B at distance $n-2$ from x such that $\pi_{F_i(x)}(y)$ coincides with the point $Q \cap B_i$. Then $\pi_{F_i(y)}(Q)$ is a quad meeting B in a point of $\Gamma_{n-2}(x)$ and C, D and E in lines.

The proposition now readily follows. ■

5.9 Glued near polygons

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two dense near polygons of diameter at least 2. If $\mathcal{A}_1$ and $\mathcal{A}_2$ satisfy certain nice properties, then it is possible to construct a so-called *glued near polygon of type* $\mathcal{A}_1 \otimes_1 \mathcal{A}_2$ (or shortly of type $\mathcal{A}_1 \otimes \mathcal{A}_2$) from $\mathcal{A}_1$ and $\mathcal{A}_2$. We refer to De Bruyn [7] for the precise details. For the purposes of the present paper, it suffices to know that a glued near polygon $\mathcal{A}$ of type $\mathcal{A}_1 \otimes \mathcal{A}_2$ satisfies the following properties: (1) there exists a partition T_i, $i \in \{1,2\}$, of $\mathcal{A}$ in convex subpolygons isomorphic to $\mathcal{A}_i$, (2) every subpolygon of T_1 intersects every subpolygon of T_2 in a line, (3) every line of $\mathcal{A}$ is contained in a convex subpolygon of $T_1 \cup T_2$, (4) $\mathcal{A}$ has diameter $\mathrm{diam}(\mathcal{A}_1) + \mathrm{diam}(\mathcal{A}_2) - 1$. Every subpolygon of $T_1 \cup T_2$ is classical in $\mathcal{A}$ and if F and F' are two subpolygons of T_i, $i \in \{1,2\}$, then the map $F' \to F, x \mapsto \pi_F(x)$ defines an isomorphism from F' to F.

Let x denote an arbitrary point of $\mathcal{A}$ and let $F_i(x)$, $i \in \{1,2\}$, denote the unique element of T_i through x. Put $L(x) := F_1(x) \cap F_2(x)$. For every convex subspace U of $\mathcal{A}$ through x, let U_i, $i \in \{1,2\}$, denote the convex subspace $F_i(x) \cap U$ of $F_i(x)$. If $L(x) \subseteq U$, then $\mathrm{diam}(U) = \mathrm{diam}(U_1) + \mathrm{diam}(U_2) - 1$ and U is a glued near polygon of type $U_1 \otimes U_2$ in case that U_1 and U_2 have diameters at least 2. If $L(x) \cap U = \{x\}$, then $\mathrm{diam}(U) = \mathrm{diam}(U_1) + \mathrm{diam}(U_2)$ and $U \cong U_1 \times U_2$ in case that U_1 and U_2 have diameters at least 1.

If V_1 and V_2 are two convex subspaces of $F_1(x)$ and $F_2(x)$, respectively, satisfying either $L(x) = V_1 \cap V_2$ or $L(x) \cap V_1 = L(x) \cap V_2 = \{x\}$, then $U := \langle V_1, V_2 \rangle$ is a convex subspace of $\mathcal{A}$ through x with $U_1 = V_1$ and $U_2 = V_2$.

The above facts have been proved in Section 3 of [7], where the convex subpolygons of glued near polygons were studied.

Proposition 5.9 *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two dense near polygons of diameter at least* 2 *and let $\mathcal{A}$ be a glued near polygon of type $\mathcal{A}_1 \otimes \mathcal{A}_2$, then $\mathcal{A}$ satisfies property (P_{de}) if and only if $\mathcal{A}_1$ and $\mathcal{A}_2$ satisfy property* (P_{de}).

Proof. Put $n = \mathrm{diam}(\mathcal{A})$. Let x denote an arbitrary point of $\mathcal{A}$, let B denote an arbitrary convex sub-$(2n-4)$-gon through x and suppose C, D and E are three convex sub-$(2n-2)$-gons through B. We will use the notations as introduced before this proposition. We distinguish two cases.

(i) Suppose that $L(x) \subseteq B$. Then we have $\mathrm{diam}(B_1) + \mathrm{diam}(B_2) = \mathrm{diam}(B)+1 = \mathrm{diam}(F_1(x))+\mathrm{diam}(F_2(x))-2$. If $\mathrm{diam}(B_1) = \mathrm{diam}(F_1(x))-1$ and $\mathrm{diam}(B_2) = \mathrm{diam}(F_2(x)) - 1$, then B is contained in precisely two convex subpolygons of diameter $n-1$, namely $\langle F_1(x), B_2\rangle$ and $\langle B_1, F_2(x)\rangle$, a contradiction. Hence, there exists an $i \in \{1,2\}$ such that $\mathrm{diam}(B_i) = \mathrm{diam}(F_i(x)) - 2$ and $B_{3-i} = F_{3-i}(x)$. As in Proposition 5.8, one can show that D and E are adjacent vertices of $\Gamma(x, B, C)$ if and only if D_i and E_i are adjacent vertices of $\Gamma(x, B_i, C_i)$.

(ii) Suppose that $B_1 \cap L(x) = \{x\} = B_2 \cap L(x)$. Then we have $\mathrm{diam}(B_1) + \mathrm{diam}(B_2) = \mathrm{diam}(B) = \mathrm{diam}(F_1(x))+\mathrm{diam}(F_2(x))-3$. Hence, there exists an $i \in \{1,2\}$ such that $\mathrm{diam}(B_i) = \mathrm{diam}(F_i(x)) - 2$ and $\mathrm{diam}(B_{3-i}) = \mathrm{diam}(F_{3-i}(x)) - 1$. Every quad intersecting B in a point and C, D and E in lines is contained in a subpolygon of T_i. As before, one can reason that D and E are collinear points of $\Gamma(x, B, C)$ if and only if D_i and E_i are collinear points of $\Gamma(x, B_i, C_i)$.

The proposition now readily follows. ■

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